# ON AN IMPROVEMENT OF THE CONVERGENCE OF THE FOURIER METHOD

(OB ULUCHSHENII SKHODIMOSTI METODA FUR'E)

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1. It is a known fact that Krylov's method [1] may be applied to improve the convergence of a common Fourier series.

An analogous method may be proposed for the solution of boundary-value problems of the harmonic type in regions represented by generalized rectangles. We will call a region a generalized rectangle if its boundaries are coordinate lines in systems of orthogonal coordinates $\xi$, $\eta$ which admit separation of variables for Laplace's equation. In addition, we will assume that the region is finite not only in a Cartesian system $y$, $x$ but also in the system $\xi$, $\eta$. Then it may be assumed without limitation of generality that the boundary $S = S_\xi + S_\eta$ of the region $\overline{\Omega} = \Omega + S$ is given by

$$\xi = \pm \alpha \, (S_\xi), \qquad \eta = \pm \beta \, (S_\eta) \qquad (0 < \alpha, \ \beta < \infty) \tag{1}$$

It is known that under these conditions the solutions of the boundary-value problems of the first, second and third kind for the Poisson equation

$$\Delta w = \frac{1}{h^2(\xi, \eta)} \left( \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} \right) = q(X), \qquad X \in \overline{\Omega} \tag{2}$$

obtained by the Fourier method, consist in the general case of the sum of a particular solution $\Psi(X)$ of Equation (2) and two infinite series

$$w = \Psi + u, \qquad u = u_\xi + u_\eta \tag{3}$$

Both series ($u_\xi$, $u_\eta$) are harmonic functions of the two variables which satisfy the corresponding homogeneous boundary conditions along the equally-named boundaries ($S_\xi$, $S_\eta$) and which become ordinary Fourier series on the other boundaries. From these boundary conditions (for $u_\xi(\xi, \beta)$ on $S_\eta$, for $u_\eta(a, \eta)$ on $S_\xi$), transformed by separation of the particular integral $\Psi(3)$, the coefficients of the series may be found just as for the ordinary expansion of functions in Fourier series.

In the case under consideration, however, a direct application of Krylov's method is not possible, first of all because $u_\xi$ and $u_\eta$ are functions of two variables. Further, in contrast to ordinary expansions in Fourier series, the function to be expanded here is not uniquely determined, since it depends essentially on the choice of the particular integral $\Psi$. Therefore, one has to select among the multitude of particular solutions of Equation (2) a function $\Psi$ which secures the best convergence for a given boundary-value problem (not only the best decrease of the coefficients of the series, but also the least of their absolute values).

2. We will show that these requirements are fulfilled, if that particular solution is chosen which satisfies the boundary conditions on two opposite sides of the generalized rectangle (1). These two sides compose the part of the boundary (for definiteness, let it be $S_\xi$) satisfying the relation

$$\frac{|S - S_\xi|}{|S_\xi|} = \frac{|S_\eta|}{|S_\xi|} = \sigma \leqslant 1 \tag{4}$$

In what follows we will consider generalized rectangles (1) in Cartesian, polar and elliptic coordinate systems.

If the loading $q(X)$ in (2) and the boundary functions along each end of the four sides of the rectangle (1) may be sufficiently well approximated by polynomials, then, as shown in [2], one may always find a particular solution satisfying the boundary conditions on any two opposite sides of the rectangle. If the particular solution is connected with opposite long sides [$S_\xi$ in (4)], we will call it in accordance with [2] the principal part of the solution and denote it by $\Psi_0$.

We will consider simultaneously the Dirichlet and the Neumann problems. The functions, referring to the Dirichlet problem will be denoted by a plus sign as subscript, those referring to the Neumann problem with a minus sign. When there is no subscript, the functions will refer to both boundary-value problems.

For definiteness we assume $w_+$ to be symmetric, $w_-$ antisymmetric in both coordinates $\xi$ and $\eta$. We have then the boundary conditions

$$\begin{aligned} (w_+)_{\xi=\pm\alpha} &= f_{2+}(\eta), \qquad & (w_+)_{\eta=\pm\beta} &= f_{1+}(\xi) \\ \left(\pm\frac{\partial w_-}{\partial \xi}\right)_{\xi=\pm\alpha} &= \pm f_{2-}(\eta), \qquad & \left(\pm\frac{\partial w_-}{\partial \eta}\right)_{\eta=\pm\beta} &= \pm f_{1-}(\xi) \end{aligned} \tag{5}$$

Selecting the particular solution $\Psi_0$ of Equation (2) which satisfies the boundary conditions (5) along $S_\xi$ given in (1) and (4), we obtain, first of all, that in (3) $u_\eta = 0$. For $u_\xi$ we find the corresponding series

$$u_{\xi+} = \sum_{(k)} A_k^{\xi} \frac{\cosh(\lambda_\xi k\eta)}{\cosh(\lambda_\xi k\beta)} \cos(\lambda_\xi k\xi), \qquad u_{\xi-} = \sum_{(k)} B_k^{\xi} \frac{\sinh(\lambda_\xi k\eta)}{\lambda_\xi k \cosh(\lambda_\xi k\beta)} \sin(\lambda_\xi k\xi) \quad (6)$$

$$(\lambda_\xi = \pi/2\alpha, \qquad k = 2r+1)$$

Integrating along $S_\eta$ by parts, we obtain the expressions for the coefficients

$$A_k^{\xi} = \frac{4}{\pi}(-1)^{\frac{k+1}{2}} \lambda_\xi \sum_{(\nu)} \frac{M_{2\nu}^{\xi}}{(\lambda_\xi k)^{2\nu+1}}, \qquad B_k^{\xi} = \frac{4}{\pi}(-1)^{\frac{k+1}{2}} \lambda_\xi \sum_{(\nu)} \frac{M_{2\nu+1}^{\xi}}{(\lambda_\xi k)^{2\nu+2}} \quad (7)$$

Here $M_{2\nu}^{\xi}$, $M_{2\nu+1}^{\xi}$ correspond to the discontinuities of the functions in Krylov's method [1] and they are expressed simply by the differential operation at the corners $(\alpha, \beta)$ of $f$ in (5) and

$$q^* \equiv qh^2 \quad (8)$$

where $q$ is the loading, $h$ is the Lamé coefficient (2). By the law of formation of $\Psi_0$ [2] we have

$$M_0^{\xi} = (-f_{1+} + f_{2+})_{\alpha,\beta}, \qquad M_1^{\xi} = \left(-\frac{\partial f_1}{\partial \xi} + \frac{\partial f_{2-}}{\partial \eta}\right)_{\alpha,\beta} \quad (9)$$

$$M_{2\nu}^{\xi} = \left((-1)^{\nu+1} \frac{\partial^{2\nu} f_{1+}}{\partial \xi^{2\nu}} + \frac{\partial^{2\nu} f_{2+}}{\partial \eta^{2\nu}} + (-1)^{\nu}\left[\frac{\partial^{2(\nu-1)} q^*}{\partial \xi^{2(\nu-1)}} - \frac{\partial^{2(\nu-1)} q^*}{\partial \xi^{2(\nu-2)}\partial \eta^2} + \dots + \frac{\partial^{2(\nu-1)} q^*}{\partial \eta^{2(\nu-1)}}\right]\right)_{\alpha,\beta}$$

$$M_{2\nu+1}^{\xi} = \left((-1)^{\nu+1} \frac{\partial^{2\nu+1} f_{1-}}{\partial \xi^{2\nu+1}} + \frac{\partial^{2\nu+1} f_{2-}}{\partial \eta^{2\nu+1}} + (-1)^{\nu}\left[\frac{\partial^{2\nu} q^*}{\partial \xi^{2\nu-1}\partial \eta} - \frac{\partial^{2\nu} q^*}{\partial \xi^{2\nu-3}\partial \eta^3} + \dots + \frac{\partial^{2\nu} q^*}{\partial \xi\, \partial \eta^{2\nu-1}}\right]\right)_{\alpha,\beta} \quad (10)$$

3. Thus, it is seen that the method leads only to a single series $u = u_\xi$.

Further, it is seen from (7), (9) and (10) that the method of selection of the principal part of the solution $\Psi_0$ achieves in a unique manner the best convergence of the coefficients, determined by a given boundary-value problem. In fact, if for $f$ there are fulfilled the conditions of continuity (absence of discontinuities in $f_+$ and in the first derivatives of $f_-$ at the corners), then $M_0^{\xi} = M_1^{\xi} = 0$ and $A_k^{\xi}$ and $B_k^{\xi}$ decrease not slower than $k^{-3}$ and $k^{-4}$, respectively. In order to use an arbitrary $\Psi$ in the Fourier method one must, for the achievement of the same convergence, apply artificial examples, since it is necessary to convert to zero at the corners not linear combinations in (9), but each of the terms in (9) separately. Further, it may occur that

$$M_2^{\xi} = \left(\frac{\partial^2 f_{1+}}{\partial \xi^2} + \frac{\partial^2 f_{2+}}{\partial \eta^2} - q^*\right)_{\alpha,\beta} = 0, \qquad M_3^{\xi} = \left(\frac{\partial^3 f_{1-}}{\partial \xi^3} + \frac{\partial^3 f_{2-}}{\partial \eta^3} - \frac{\partial^2 q^*}{\partial \xi\, \partial \eta}\right)_{\alpha,\beta} = 0 \quad (11)$$

i.e. the decrease of $A_k^{\xi}$ and $B_k^{\xi}$ is even stronger (as $k^{-5}$ and $k^{-6}$, respectively) etc. (for $M_3$, $M_5$, etc.). In spite of the fact that the boundary-value problem determines a very strong decrease in the coefficients, realisable for the selection of $\Psi_0$, this is easily seen not to be so for an arbitrary $\Psi$.

All that has been said so far applies essentially also for the particular solution connected with the two opposite sides $S_\eta$. One has to replace in (6), (7), (9), (10) and (11) the index $\xi$ by $\eta$, to exchange $\alpha$ and $\beta$, and it follows from (10) that

$$M^{\eta}_{2\nu,\,2\nu+1} = (-1)^{\nu+1} M^{\xi}_{2\nu,\,2\nu+1} \tag{12}$$

However, only by isolating $\Psi_0$ is there secured not only the best decrease of the coefficients but also the smallest modulus of the remaining series. It may be shown that for given accuracy

$$\varepsilon_0 = \frac{\| w - w_n \|}{\| w \|} = \frac{\| u - u_n \|}{\| u \|} \frac{\| u \|}{\| w \|} \tag{13}$$

the number of the retained terms $n_\xi$ in isolating $\Psi_0$ is smaller than the corresponding number of terms $n_\eta$ (connected with the short sides), the smaller $\sigma$ in (4). One has the estimate

$$n_\xi \approx n_\eta C \sigma \tag{14}$$

where $C$ is close to unity. In polar coordinates ($\xi = \log r$, $\alpha^2 = \rho$)

$$C \approx \left\{1 + \frac{1}{3}\left(\frac{\rho-1}{\rho+1}\right)^2 + \frac{1}{5}\left(\frac{\rho-1}{\rho+1}\right)^4 + \dots\right\}\left\{1 + \frac{1}{2}\left(\frac{\rho-1}{\rho}\right) + \frac{1}{3}\left(\frac{\rho-1}{\rho}\right)^2 + \dots\right\}^{-1/r} \tag{15}$$

where for the Dirichlet problem $r \geqslant 6$ for $M_0 = 0$, for the Neumann problem $r = 6$ for $M_1 \neq 0$, $r = 10$ for $M_1 = 0$ etc.

It may likewise be shown that for arbitrary $\Psi$ (even securing the same convergence of coefficients as $\Psi_0$) one has always $n \geqslant n_\xi$. For the rectangle in Cartesian coordinates one may, in addition, obtain an estimate for the mean value $n_{\text{mean}}$ (for all arbitrary $\Psi$, securing the same convergence of the coefficients as $\Psi_0$):

$$n > n_\xi / \sigma = n_\eta, \qquad \sigma < 1 \tag{16}$$

In passing, we note that there is a unique exception from all the stated results for rectangles in Cartesian coordinates $x$, $y$. If the Dirichlet problem is symmetric with respect to $(x/\alpha,\ y/\beta)$ and $M_0 = 0$, $M_2 \neq 0$, then $A_k^{\xi} = A_k^{\eta}$, i.e. the two sequences of coefficients degenerate into one, when the particular solution is connected with the ellipse with semi-axes $(\sqrt{2}\,\alpha, \sqrt{2}\,\beta)$. In this case one has to select as principal part $\Psi_0$ of the solution the particular solution connected with the stated ellipse. As has been shown in [2], this may always be done by reducing beforehand the boundary conditions to their homogeneous form and by constructing the particular solution satisfying the homogeneous conditions over the stated ellipse.

Returning to $\Psi_0$ connected with $S_\xi$, we will show that the method permits, in the same way as Krylov's method [1], a further improvement in the convergence of the series. In fact, it is seen from (9)-(12) that, in order to obtain artificially $M_2 = 0$, it is sufficient to know the solution of the simple problem with loading $q^* = 1$ and $f_1 = f_2 = 0$, while for $M_3 = 0$ one needs that for $q^* = \xi\eta$ and $f_1 = f_2 = 0$, for $M_4 = 0$ one needs $q^* = \xi^2$ etc. The solutions of these very simple problems superimposed may easily be found by the same method of selection of the principal part of the solution.

We note that the described method also permits us to find coefficients $A_k$, $B_k$ of the series, as is seen from (6) and (7) without integration, by differentiation of the loading and boundary functions at the corners. In those cases when the series in $\nu$ in (7) converge, this is true for all $A_k$, $B_k$. In those cases when the series in $\nu$ may diverge (if $q$ and $f$ are given in the form of polynomials of higher order $l$ in $x$, $y$ and the rectangle is in coordinates $\xi$, $\eta$) the evaluation of the coefficients in (7) is also possible, but only beginning approximately with $k > [l/\lambda_\xi]$.

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